DECOMPOSITIONS OF THE INCIDENCE MATRICES OF UNDIRECTED GRAPHS

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MOTIVATION

- Matrix decomposition is a linear algebra tool which has been used to analyze many high-dimensional data sets in real-world applications
- Graphs can model diverse phenomena, from social networks to computer architecture
- Is it possible to expand the use of matrix decomposition into the pure field of graphs, and thereby open additional possible applications?

GRAPH PRODUCTS

Using two results from linear algebra, we show that the rank-1 approximation behaves well under the Cartesian product of graphs. Suppose we have the graphs G and H with adjacency matrices A_G and A_H. We can make use of two facts from linear algebra:

Lemma 1 ([1]). *Let λ_i, $1 \leq i \leq n$ represent the eigenvalues of G and μ_j, $1 \leq j \leq m$ represent the eigenvalues of H. Then the eigenvalues of $G \square H$ are given by $\lambda_i + \mu_j$.*

Lemma 2 ([1]). *If $\vec{v}_i$ is the eigenvector of A_G corresponding to λ_i and $\vec{w}_j$ is the eigenvector of A_H corresponding to μ_j, then $\vec{v}_i \otimes \vec{w}_j$ is the eigenvector of $A_{G \square H}$ corresponding to $\lambda_i + \mu_j$.*

These lemmas, along with a reforumaltion of the cartesian product, yield the following result:

Theorem 1. *Let A_G, D_G, and $Y_G = Z_G Z_G^T$ represent the adjacency, degree, and signless Laplacian matrices of G and H, respectively. Further, let $\lambda_i(Y_G)$ and $\lambda_j(Y_H)$ be the eigenvalues of Y_G and Y_H with $\lambda_1(Y_G) \geq \ldots \geq \lambda_n(Y_G)$ and $\lambda_1(Y_H) \geq \ldots \geq \lambda_n(Y_H)$. Finally, let $\vec{v}_i(Y_G)$ and $\vec{v}_j(Y_H)$ be the eigenvectors corresponding to $\lambda_i(Y_G)$ and $\lambda_i(Y_H)$, respectively. Then*

- *The eigenvectors $\lambda_{i+j}(Y_{G \square H})$ are all given by $\lambda_i(Y_G) + \lambda_j(Y_H)$ for all combinations of $1 \leq i \leq n$ and $1 \leq j \leq m$.*
- *The eigenvector corresponding to $\lambda_{i+j}(Y_{G \square H})$, $\vec{v}_{i+j}(Y_{G \square H})$, is given by $\vec{v}_i(Y_G) \otimes \vec{v}_j(Y_H)$.*

COMPUTATIONAL RESULTS

Graph	Dimensions	Pseudocode (conditions for $Z_{ij} = 1$)
Path	$n \times n - 1$	$i = j$ or $i = j + 1$
Cycle	$n \times n$	$i = j$ or $i - 1 \equiv j \mod n$
Star	$n \times n - 1$	$i = 1$ or $i = j + 1$
Wheel	$n \times 2(n-1)$	$\begin{bmatrix} \begin{bmatrix} n \text{ star} \end{bmatrix} & \begin{bmatrix} 0 \ldots 0 \\ n-1 \text{ cycle} \end{bmatrix} \end{bmatrix}$
Complete	$n \times n(n-1)/2$	columns are permutations of $\{1, 1, 0, \ldots, 0\}$
Complete bipartite	$n \times n^2/4$	$i - n \equiv j \mod n$ or $i - 1 = (j-1) \backslash n$
Prism	$n \times \frac{3}{2}n$	$\begin{bmatrix} \begin{bmatrix} n \text{ cycle} \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & I \\ \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} n \text{ cycle} \end{bmatrix} & I \end{bmatrix}$
Tree	$n \times n - 1$	$j = i - 1$ or $i = (j + s - 1) \backslash s$

Table 1: Summary of classes of graphs for which incidence matrices were generated

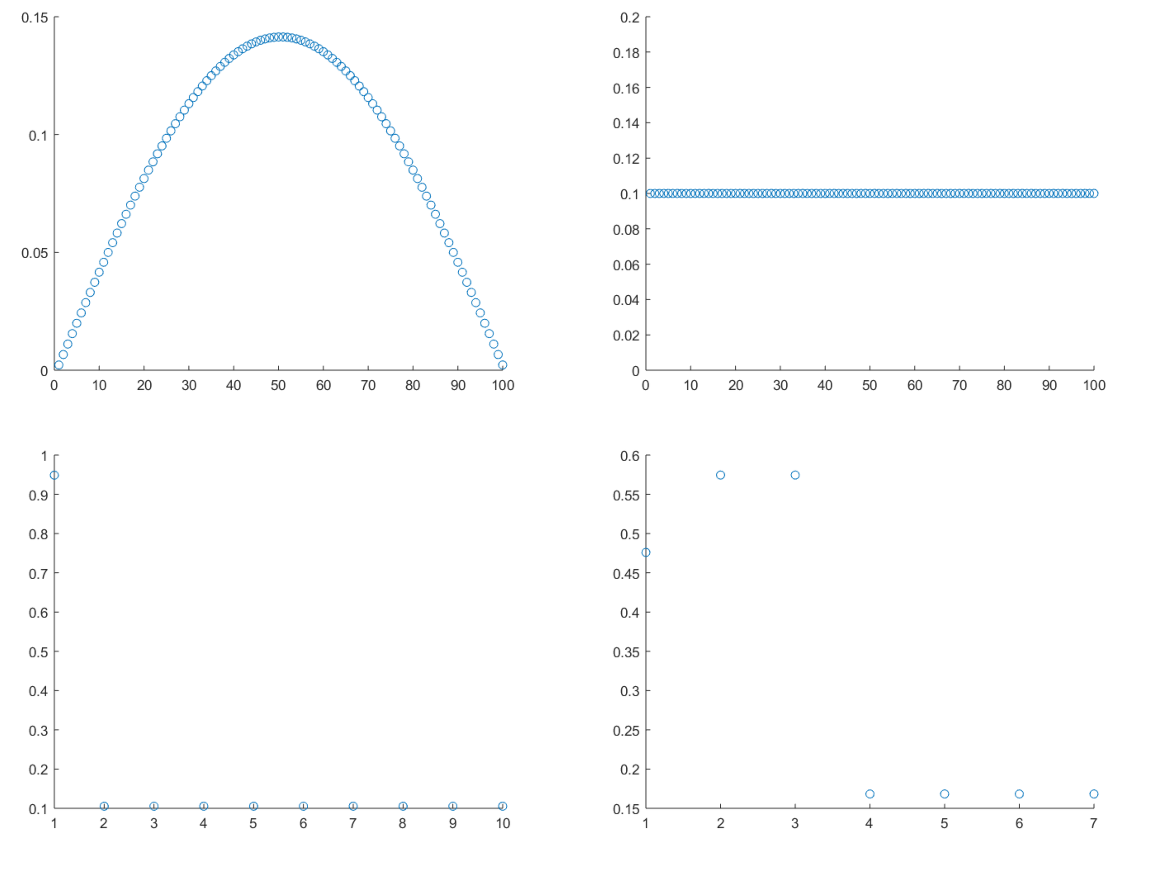

Figure 1: Plots of rank-1 approximation entries for, respectively, a 100-vertex path, a 100-vertex cycle, a 10-vertex star, and a 7-vertex binary tree

SIMPLE CLASSES OF GRAPHS

We develop and then prove explicit results for the rank-1 approximations of several simple classes of graphs. In each case, the proof begins with the general structure of the incidence matrix given in table 1, then uses matrix manipulation techniques to obtain an explicit or recursive formula for the determinant, and finally, uses this to proove the contents of the rank-1 approximation.

- **Cycle, C_n:** 4 is the largest eigenvalue of ZZ^T for a cycle of any size. The rank-1 approximation consists of $\vec{1}$.
- **Star, S_{n-1}:** n is the largest eigenvalue of ZZ^T for a star of any size. The rank-1 approximation consists of vectors of the form $\{n-1, 1, 1, \ldots, 1, 1\}$
- **Complete graph, K_n:** $2n-2$ is the largest eigenvalue of ZZ^T for a complete graph of any size. The rank-1 approximation consists of .

TREE EXAMPLES

Computational results suggest that patterns emerge also in some complex classes of graphs; these examples support the conclusion that the rank-1 approximation is related to vertex centrality.

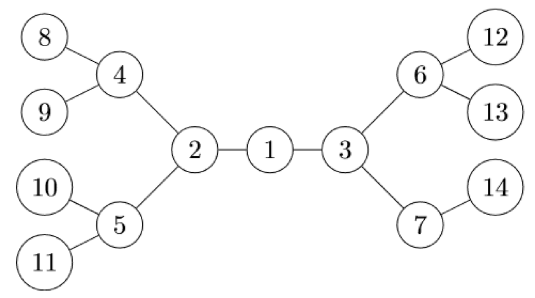

Figure 2: A tree with $n = 14$ and $s = 2$; each vertex has 2 branches.

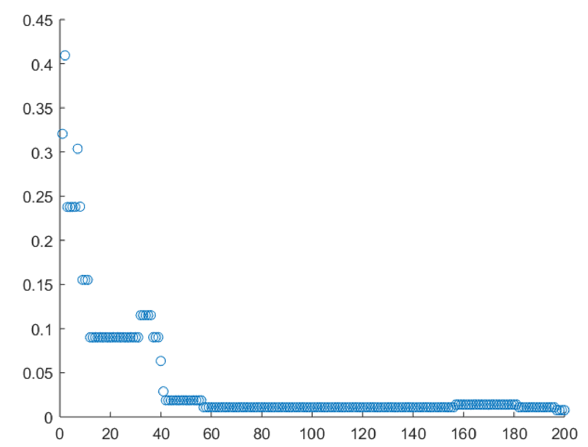

Figure 3: Rank-1 approximation entries for a tree with $n = 200$, $s = 5$

VERTEX CENTRALITY

Computational examples and theoretical results suggest that the rank-1 approximation gives information about the centrality of vertices within graphs. To formalize this notion, we find bounds for the difference between the rank-1 approximation and the already well-known eigenvector centrality.

Davis-Kahan $\sin(\theta)$ Theorem[2]: Let Σ and $\hat{\Sigma}$ be symmetric $n \times n$ matrices with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and eigenvectors $v_1, v_2, \ldots, v_n$. Then for all j, provided that $\hat{v}_j^T v_j \geq 0$,

$$\sin \Theta(\hat{v}, v) \leq \frac{2\left|\left|\hat{\Sigma} - \Sigma\right|\right|_{op}}{\min\left(\left|\hat{\lambda}_{j-1} - \lambda_j\right|, \left|\hat{\lambda}_{j+1} - \lambda_j\right|\right)} \quad (1)$$

In our case, call $\hat{\Sigma} = ZZ^T$ and $\Sigma = A$. Recall that since ZZ^T is the signless Laplacian matrix,

$$ZZ^T = A + D \quad (2)$$

so $D = \hat{\Sigma} - \Sigma$. Under these equivalences, we have $\hat{v}_j = v_j\left(ZZ^T\right)$, the eigenvector corresponding to the j^{th} largest eigenvalue of ZZ^T, and $v_j = v_j(A)$, the eigenvector corresponding to the j^{th} largest eigenvalue of A. Then, simplifying, we obtain

$$\sin \Theta\left(v_1\left(ZZ^T\right), v_1(A)\right) \leq \frac{2\,||D||_{op}}{\left|\lambda_2\left(ZZ^T\right) - \lambda_1(A)\right|} \quad (3)$$

Equation 3 shows that the angular difference between $v_1\left(ZZ^T\right)$ (the rank-1 approximation) and $v_1(A)$ (the eigenvector centrality measure) is bounded above. It therefore confirms that the rank-1 approximation measures vertex centrality, and provides a result which is only boundedly different from the already-known measure given by the adjacency matrix. In addition, equation 3 shows that the relationship depends on the degrees of vertices in the underlying graphs; an increase in vertex degrees leads to a less stringent bound.

REFERENCES

[1] R. B. Bapat S. Barik and S. Pati. On the laplacian spectra of product graphs. *Applicable Analysis and Discrete Mathematics*, 9:39–58, 2015.

[2] T. Wang Y. Yu and R. J. Samworth. A useful variant of the davis-kahan theorem for statisticians. *Biometrika*, 102:315–323, 2015.

FUTURE RESEARCH

Both the computational work and theoretical results suggest several directions for future research. In particular:

- How does the rank-1 approximation behave under graph operations other than the Cartesian product?
- How much about a graph can we tell from its rank-1 approximation? There seem to be different graphs with the same approximation. What about rank $r > 1$ approximations?
- Can the bounds imposed by the Davis-Kahan Theorem be improved or differently applied?